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Calculation of the $\beta^+\beta^+$, β^+/EC and EC/EC half-lives for ^{106}Cd with the second quasi random phase approximation method

S. Stoica[1,a] and H.V. Klapdor-Kleingrothaus[2]

[1] Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering, P.O. Box MG-6, 76900 Bucharest, Romania
[2] Max-Planck-Institut für Kernphysik, W-6900 Heidelberg, Germany

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Abstract. The double-beta decay matrix elements and half-lives for ^{106}Cd are computed with the second quasi random phase approximation (SQRPA) method and using two single-particle (s.p.) bases. For the neutrino-emitting decay modes the two-positron emission ($\beta^+\beta^+$), the positron emission/electron capture (β^+/EC) and the double electron capture (EC/EC) processes are treated. It was found that the nuclear matrix elements (NME) display a strong dependence on the strength of the particle-particle interaction and an important contribution to the decay amplitude is coming from the 1^+ ground state of the intermediate nucleus ^{106}Ag. Their values depend weakly on the s.p. basis used. For both bases the deviations from the Ikeda sum rule are only within 2-3%. We got half-lives for the β^+/EC of the order of $\sim 10^{21}$ y which is not far from the actual experimental limits. For the neutrinoless $\beta^+\beta^+$ and β^+/EC decay modes the NME relevant both for the mass mechanism and the right-handed (RH) currents were calculated. They are found to be slightly larger than those obtained in our previous calculations (M. Hirsch, K. Muto, T. Oda, H.V. Klapdor-Kleingrothaus, Z. Phys. A **334**, 151 (1994)). Using the value of the neutrino mass parameter extracted from the recently reported first experimental evidence of the neutrinoless decay mode (*i.e.* 0.39 eV) (H.V. Klapdor-Kleingrothaus, A. Dietz, H.I. Harney, I.V. Krivosheina, hep-ph/0201231; Mod. Phys. Lett. A **16**, 2409 (2001); H.V. Klapdor-Kleingrothaus, A. Dietz, I.V. Krivosheina, Part. Nucl. Lett. **110**, 57 (2002); Found. Phys. **32**, 1181 (2002)), we got half-lives of $\sim 10^{28}$ y and 10^{27} y, for the $0\nu\beta^+\beta^+$ and $0\nu\beta^+$/EC processes, respectively. An experimental investigation of these decays could be useful for testing the importance of the right-handed current mechanism to the occurrence of neutrinoless $\beta\beta$ decay.

PACS. 21.60.Jz Hartree-Fock and random-phase approximations – 23.40.Hc Relation with nuclear matrix elements and nuclear structure – 23.40.Bw Weak-interaction and lepton (including neutrino) aspects

1 Introduction

The double-beta ($\beta\beta$) decay process is still receiving much attention since its study could provide us with essential information about the neutrino properties, structure of the weak interaction as well as about physics beyond the Standard Model (SM) [1]. One of the challenging problems related to this process is the accurate calculation of the nuclear matrix elements (NME) which appear in the half-lives formulae of the different $\beta\beta$ decay modes. The most extensively used approach for their computation was the proton-neutron quasi random phase approximation (pnQRPA), since with this method one could reproduce the experimental half-lives of the two-neutrino double-beta ($2\nu\beta\beta$) decay mode for a large number of nuclei. However, calculated with this method the NME manifest a large sensitivity to the renormalization of the particle-particle strength in the 1^+ channel ($g_{\rm pp}$). To overcome this drawback, several improvements of pnQRPA have been tried. For the whole history of these developments and an up-to-date situation of the calculations we suggest the reader to consult some comprehensive reviews like [2–6, 1].

The most successful developments in getting more stable results for the NME have been proved to be the extensions of the pnQRPA method beyond the basic approximation that it assumes, namely the quasi boson approximation (QBA). On this line two different approaches have been developed. First, the idea that particle-like correlations at the RPA level should also contribute, besides the particle-unlike ones, to the $\beta\beta$ decay process was exploited in [7]. In those references the first-order corrections

[a] e-mail: stoica@ifin.nipne.ro

beyond the QBA have been taken into account by a boson expansion of the quasiparticle pair operators which are involved in the pnQRPA formalism. Several years later, the idea of restoring (partially) the Pauli exclusion principle which is violated within pnQRPA, leads to the renormalization pnQRPA (RQRPA) approach, where higher-order corrections to pnQRPA have been introduced by taking into account in an approximative way the next terms in the boson commutator relations of the pair quasiparticle operators [8].

With the pnQRPA method and its extensions a lot of calculations of the NME for $\beta^-\beta^-$ decay have been performed, including two- and zero-neutrino modes and transitions to ground and excited final states [9–25]. By contrast, the $\beta^+\beta^+$, β^+/EC and the EC/EC decay processes have received much less attention. This could be explained by the relatively larger available kinetic energies of several nuclei which undergo a $\beta^-\beta^-$ decay, and by the Coulomb attraction between the outgoing electrons and nucleus which favour shorter half-lives for this decay process and makes it experimentally more attractive. This is why many set-ups for measuring $\beta^-\beta^-$ decay have been developed and the most stringent half-lives and neutrino mass limits come from the $0\nu\beta^-\beta^-$ measurements.

However, the β^+-type decays represent another challenge for the calculation of the nuclear-structure part of the weak-interaction processes. In the recent past, the interest for these decays increased due to some progress in the experimental set-ups. On the other hand, there are recent evaluations of NME and half-lives for these modes [8, 26–29]. The authors of those references have used the pnQRPA and RQRPA methods and found values of half-lives which could be accessible in the near future for experiments. A common feature of those calculations is that the NME are highly dependent on the model parameters, which affects the accuracy of the calculations. This feature is particular true in the case of ^{106}Cd. This is why it is worthwhile to perform calculations with different approaches.

In this paper we calculate the NME and half-lives for the two-neutrino $\beta^+\beta^+$, β^+/EC and EC/EC and for neutrinoless $\beta^+\beta^+$ and β^+/EC decay processes with the SQRPA method in the case of ^{106}Cd. This method was recently used extensively in $\beta^-\beta^-$ calculations [23–25], but is used here for the first time for β^+-type decay modes. Our calculation is also motivated by the recent experimental investigations of this isotope [29–31], where limits of half-lives of the order of 10^{19}–10^{20} y were reached.

Our goal is to study the suppression of the NME and their dependence on the model parameters and on the size of the s.p. model space used. On the other hand, in ref. [28] it was shown that with observation of the $0\nu\beta\beta$ mode, the study of positron-emitting decay modes might decide which mechanism would give the main contribution to this process: the mass mechanism or the existence of right-handed (RH) weak currents. In this respect, it would be of interest to see which coefficients involved in the formulae for the half-lives of these processes are more important, and this is another goal of our work.

The paper is organized as follows: in sect. 2 we give details on the formalism we use for computing the 2ν and 0ν NME and half-lives. In sect. 3 we present the numerical calculations and discussions of the results and in sect. 4 we end up with conclusions. In appendix A we give detailed expressions of some quantities used in our calculations.

2 Formalism

We give in this section the relevent formulae used in the calculations. In the SQRPA the intermediate odd-odd nucleus is described by states of one- and two-boson type which are obtained by the action of the proton-neutron (pn) and proton-proton (pp) and neutron-neutron (nn) QRPA phonon operators onto the vacua of the initial (i) and final (f) nuclei participating in the $\beta\beta$ decay:

$$\Gamma^+_{1\mu}(k)|0\rangle_{i,f}\,,\;;\qquad \left[\Gamma^+_1(k_1)\Gamma^+_2(k_2)\right]_{1\mu}|0\rangle_{i,f}\,. \tag{2.1}$$

The $\Gamma^\dagger$-operators are defined as follows:

$$\Gamma^\dagger_{1\mu}(k)=\sum_{l=(j_p,j_n)}\left[X^1_k(l)A^+_{1\mu}(l)+Y^1_k(l)\tilde{A}_{1\mu}(l)\right],$$
$$\Gamma^\dagger_{2\mu}(k')=\sum_{l'=(j_p,j'_p;j_n,j'_n)}\left[X^2_{k'}(l')A^+_{2\mu}(l')+Y^2_{k'}(l')\tilde{A}_{2\mu}(l')\right], \tag{2.2}$$

such that

$$\Gamma_{J\mu}(k)|0\rangle_{i,f}=0\,,\qquad J=1,2\,.$$

X and Y are the forward- and backward-going QRPA amplitudes labeled with the indice (1) for the pn mode and indice (2) for the pp and nn modes; $l\equiv(p,n)$ denotes a pn pair, $l'\equiv(p,p),(n,n)$ denotes pp or nn pairs and k, k' label the positive solutions of the pnQRPA, QRPA equations, respectively.

$|(1_{k_1}2_{k_2})\rangle$ in (2.1) are corrections to the QRPA 1^+ wave functions built with both dipole (pn) and quadrupole (pp and nn) operators and their physical meanings that the 1^+ states in the intermediate odd-odd nucleus can also be built from excited 2^+ states of the parent nucleus.

The A, $A^\dagger$ are the bi-fermion quasiparticle operators coupled to angular momentum $J=1,2$ and projection μ:

$$A^\dagger_{J\mu}(l)=\sum_{m_p,m_n}C^{j_pj_nJ}_{m_pm_n\mu}a^\dagger_{j_pm_p}a^\dagger_{j_nm_n}\,;$$
$$\tilde{A}_{J\mu}=(-)^{J-\mu}A_{J,-\mu}\,. \tag{2.3}$$

In the QBA the operators $A^\dagger$, A as well as the phonon operators $\Gamma^\dagger$, Γ fulfill the boson-type commutator relations.

In the formalism we also need the following bi-fermion density-type operators:

$$B^\dagger_{1\mu}(l)=\sum_{m_p,m_n}C^{j_pj_nJ}_{m_p-m_n\mu}a^\dagger_{j_pm_p}a_{j_nm_n}(-)^{j_n-m_n}\,;$$
$$\tilde{B}_{1\mu}(l)=(-)^{1-\mu}B_{1\mu}(l)\,. \tag{2.4}$$

$$M_{\rm GT} = \frac{\langle 0_i||\beta^-||1_{k_1}\rangle\langle 1_{k_1}|1_{k_1'}\rangle\langle 1_{k_1'}||\beta^-||0_f\rangle}{m_ec^2+Q_{\beta\beta}/2+\Delta E_1} + \frac{\langle 0_i||\beta^-||(1_{k_1}2_{k_2})1\rangle\langle(1_{k_1}2_{k_2})1|(1_{k_1'}2_{k_2'})1\rangle\langle(1_{k_1'}2_{k_2'})1||\beta^-||0_f\rangle}{m_ec^2+Q_{\beta\beta}/2+\Delta E_2} \tag{2.12}$$

In the particle representation the transition operators $\beta^\pm$ are defined as follows:

$$\beta^-_\mu(l) = \sum_{m_pm_n}\langle j_pm_p|\sigma_\mu|j_nm_n\rangle c^\dagger_{j_pm_p}c_{j_nm_n}\,;$$
$$\beta^+_\mu(l) = (-)^\mu\left(\beta^-_{-\mu}(l)\right)^\dagger, \tag{2.5}$$

where σ_μ denotes the μ-th spherical component of the Pauli spin operator. Their expressions in the quasiparticle representation read [5]

$$\beta^-_\mu(l) = \theta_lA^\dagger_{1\mu}(l)+\bar\theta_l\tilde A_{1\mu}(l)+\eta_lB^\dagger_{1\mu}(l)+\bar\eta_l\tilde B_{1\mu}(l)\,,$$
$$\beta^+_\mu(l) = -\left(\bar\theta_lA^\dagger_{1\mu}(l)+\theta_l\tilde A_{1\mu}(l)+\bar\eta_lB^\dagger_{1\mu}(l)+\eta_l\tilde B_{1\mu}(l)\right), \tag{2.6}$$

where the following notations are used:

$$\theta_l = \frac{\hat j_p}{\sqrt3}\langle j_p||\sigma||j_n\rangle U_pV_n\,,$$
$$\bar\theta_l = \frac{\hat j_p}{\sqrt3}\langle j_p||\sigma||j_n\rangle U_nV_p;\ \hat j = \sqrt{2j+1}\,;$$
$$\eta_l = \frac{\hat j_p}{\sqrt3}\langle j_p||\sigma||j_n\rangle U_pU_n\,,$$
$$\bar\eta_l = \frac{\hat j_p}{\sqrt3}\langle j_p||\sigma||j_n\rangle V_pV_n\,. \tag{2.7}$$

In the SQRPA method the higher-order corrections to the pnQRPA are introduced by expanding the bi-fermion operators $A^\dagger$, A, $B^\dagger$, B into a series of boson operators. Up to two boson terms one gets [7]:

$$A^\dagger_{1\mu}(l) = \sum_k\left(A^{(1,0)}_{k_1}\Gamma^+_{1\mu}(k)+A^{(0,1)}_{k_1}\tilde\Gamma^+_{1\mu}(k)\right), \tag{2.8}$$

$$B^\dagger_{1\mu}(l) = \sum_{k_1k_2}\Big(B^{(2,0)}_{k_1k_2}(l)[\Gamma^\dagger_1(k_1)\Gamma^\dagger_2(k_2)]_{1\mu} + B^{(0,2)}_{k_1k_2}(l)[\Gamma_1(k_1)\Gamma_2(k_2)]_{1\mu}\Big)\,. \tag{2.9}$$

The boson expansion coefficients $A^{(1,0)}$, $A^{(0,1)}$, $B^{(2,0)}$, $B^{(0,2)}$ are determined so that eqs. (2.8), (2.9) are also valid for the corresponding ME in the boson basis. Their expressions are given in appendix A of ref. [7].

For consistency, using (2.8), (2.9), we also derived the expressions of the $\beta^\pm$-operators in the same order of approximation as the phonon operators. Their complete expressions can be found in ref. [7]. Further, one can calculate the matrix elements of the single $\beta^\pm$-operators between the states of our model space (2.1). We give here that part of the expressions which contributes to the g.s.-to-g.s. transitions:

$$\langle 0||\beta^-||1_k\rangle = \left(Y^1_k(j_pj_n)\theta_l - X^1_k(j_pj_n)\bar\theta_l\right), \tag{2.10a}$$
$$\langle 0||\beta^+||1_k\rangle = \sqrt3\left(Y^1_k(j_pj_n)\bar\theta_l - X^1_k(j_pj_n)\theta_l\right), \tag{2.10b}$$

$$\langle 0||\beta^-||(1_{k_1}2_{k_2})1\rangle = 2\sqrt{15}\,\Big(Z^{112}_{p'pn}X^2_{k_2}(j_pj_{p'})Y^1_{k_1}(j_{p'}j_n) + Z^{112}_{n'np}Y^2_{k_2}(j_{n'}j_n)X^1_{k_1}(j_pj_{n'})\Big)\,, \tag{2.11a}$$

$$\langle 0||\beta^+||(1_{k_1}2_{k_2})1\rangle = 6\sqrt5\,\Big(Z^{112}_{p'pn}Y^2_{k_2}(j_pj_{p'})X^1_{k_1}(j_{p'}j_n) + Z^{112}_{n'np}X^2_{k_2}(j_{n'}j_n)Y^1_{k_1}(j_pj_{n'})\Big)\,, \tag{2.11b}$$

where

$$Z^{abc}_{def} = (-)^{d-f}W(adbe;fc)\,.$$

We remark that the expressions (2.11a), (2.11b) represent the corrections within our formalism to the QRPA usual expressions of the $\beta\pm$ transition amplitudes.

For the $\beta^+\beta^+$ decay and $2\nu\beta\beta$ decay mode we obtain for the Gamow-Teller matrix elements the following expression:

see eq. (2.12) *above*

with

$$\Delta E_1 = \frac12\sum_{\alpha=i,f}\left(E^\alpha_1(k)-E^\alpha_1(1)\right)+E^{\rm exp}_1$$
$$\Delta E_2 = \frac12\sum_{\alpha=i,f}\left(E^\alpha_1(k_1)+E^\alpha_2(k_2)-E^\alpha_1(1)\right)+E^{\rm exp}_1\,,$$

where $E^\alpha_1(k)$ and $E^\alpha_2(k)$ are the pnQRPA and QRPA energies and $E^{\rm exp}_1$ are the first 1^+ experimental energies in the intermediate nucleus. The expressions of the overlap factors between the virtual intermediate states are given in appendix A. Thus, in SQRPA the expression of the NME for the two-neutrino decay mode contains, besides the usual pnQRPA term (the first term in (2.12)), a higher-order contribution in which two-body particle-like correlations are also involved. This contribution is introduced in a consistent way, namely it comes from both the enlargement of our model space (from one- to two-boson states) as compared with the pnQRPA model space and the expressions of the β^- transition operators (see also eq. (3.7) of ref. [7]) which also contain terms beyond their quasiparticle representation (2.6).

The $\beta^+\beta^+$ half-lives expressions can be written as [28]

$$\left[T_{1/2}^{2\nu,a}\right]^{-1} = F^{2\nu,a}|M_{\mathrm{GT}}^{2\nu}|^2 , \tag{2.13}$$

$$\left[T_{1/2}^{0\nu,a}\right]^{-1} = C_{mm}^a\left(\frac{\langle m_\nu\rangle}{m_e}\right)^2 + C_{\eta\eta}^a\langle\eta\rangle^2 + C_{\lambda\lambda}^a\langle\lambda\rangle^2 + C_{m\eta}^a\frac{\langle m_\nu\rangle}{m_e}\langle\eta\rangle + C_{m\lambda}^a\frac{\langle m_\nu\rangle}{m_e}\langle\lambda\rangle + C_{\eta\lambda}^a\langle\eta\rangle\langle\lambda\rangle , \tag{2.14}$$

where a denotes the different $\beta^+\beta^+$, β^+/EC and EC/EC decay modes and $\langle m_\nu\rangle$, $\langle\eta\rangle$ and $\langle\lambda\rangle$ are the effective neutrino mass and right-handed parameters. The coefficients C_{xy} are defined as follows [10]:

$$C_{mm}^a = F_1^a\left(M_{\mathrm{GT}} - M_F\right)^2 , \tag{2.15a}$$
$$C_{m\lambda}^a = \left(M_{\mathrm{GT}} - M_F\right)\left(-S_e M_{2-}F_3^a + M_{1+}F_4^a\right) , \tag{2.15b}$$
$$C_{m\eta}^a = \left(M_{\mathrm{GT}} - M_F\right)\left[S_e M_{2+}F_3^a - M_{1-}F_4^a - S_\beta\left(M_P F_5^a - S_e M_R F_6^a\right)\right] , \tag{2.15c}$$
$$C_{\lambda\lambda}^a = M_{2-}^2F_2^a - \frac{1}{9}f_e\left(2S_e M_{1+}M_{2-}F_3^a - M_{1+}^2F_4^a\right) , \tag{2.15d}$$
$$C_{\eta\eta}^a = M_{2+}^2F_2^a - \frac{1}{9}f_e\left(2S_e M_{1-}M_{2+}F_3^a - M_{1-}^2F_4^a\right) - S_e M_P M_R F_7^a + M_P^2F_8^a + M_R^2F_9^a , \tag{2.15e}$$
$$C_{\lambda\eta}^a = -2\left(M_{2+}M_{2-}F_2^a - \frac{1}{9}f_e\left[S_e\left(M_{1+}M_{2+} + M_{1-}M_{2-}\right)F_3^a - M_{1+}M_{1-}F_4^a\right]\right) , \tag{2.15f}$$

where

$$M_{1\pm} = M_{\mathrm{GT}q} - 6M_T \pm 3M_{Fq} ,$$
$$M_{2\pm} = M_{\mathrm{GT}\omega} \pm M_{F\omega} - \frac{1}{9}M_{1\mp} . \tag{2.16}$$

The definitions of the factors S and f, which distiguish between the different decay modes, can be found in [28] and the expressions of the nine neutrinoless matrix elements involved in (2.15) are given in appendix A. $F^{2\nu,a}$ and F_k^a are the phase space factors for the two-neutrino and neutrinoless decay modes, respectively.

We also checked in our calculations the Ikeda sum rule (ISR) [32] which must be fulfilled independently of the nuclear model used:

$$S_- - S_+ = \Sigma_m|\langle 1_m^+||\beta_m^-||0_{\mathrm{g.s.}}^+\rangle|^2 - |\langle 1_m^+||\beta_m^+||0_{\mathrm{g.s.}}^+\rangle|^2 = 3(N-Z) . \tag{2.17}$$

3 Numerical calculations and discussion of results

We calculate the NME and the half-lives for the $\beta^+\beta^+$, β^+/EC and EC/EC processes for the neutrino-emitting modes and for the $\beta^+\beta^+$ and β^+/EC processes for the neutrinoless decay modes. The process 0νEC/EC is not yet settled theoretically, since this decay mode must be accompanied by the emission of other particles for reasons of energy momentum conservation. Consequently, its decay rate has to be calculated at least by the third order of perturbation theory and is expected to be at least by four orders of magnitude smaller than the β^+/EC decay mode [33].

In the numerical calculations we follow a similar procedure that we used in our previous papers [24,25]. For the Hilbert space used to generate the s.p. basis we made two choices: i) the full (3–5) $\hbar\omega$ oscillator shells plus the orbital $i_{13/2}$ and ii) the full (2–5) $\hbar\omega$ oscillator shells plus the orbital $i_{13/2}$. From here on, we will call (s) the smaller basis i) and (l) the larger basis ii) and these indices are also used in tables and figures to distinguish between calculations performed with the two bases. The s.p. energies were obtained by solving the Schrödinger equation with a Coulomb-corrected Wood-Saxon potential with the Bohr-Mottelson parametrization [34]. The λ-pole nucleon-nucleon residual interactions were taken as Brueckner G-matrix derived from the Bonn-A one-pion exchange potential. The quasiparticle energies and the BCS occupation amplitudes were calculated by solving the HFB equations with particle-like pairing interaction, separately for the initial and final nuclei participating in the $\beta\beta$ decay, and for the two basis sets. The renormalization constants for the pp and nn pairing interactions were fixed by fitting the experimental mass differences between even and odd neighboring nuclei. In the second step of the calculation we solved the pnQRPA and pp-nn QRPA equations for the initial (^{106}Cd) and final (^{106}Pd) nuclei and for the two basis sets. In the pnQRPA numerical calculations the renormalization constants of the particle-hole residual interactions were fixed as follows: for the 1^+ channel it was fixed so that to reproduce the Gamow-Teller Giant Resonance in ^{106}Ag. For the other multipolarities they were taken 1.0 for all the multipolarities except the 2^+ one, where it was taken 0.8, since for larger values the p-h interaction in this channel is too strong, producing the collapse of the RPA procedure. The strength of the particle-particle interaction in the 1^+ channel, g_{pp}, was left as a free parameter when we studied the dependence of the NME on this parameter. However, we fixed this constant from single β decays, as is specified below, when we calculated the half-lives. In the pp-nn QRPA procedures the renormalization constants of the particle-hole interactions were fixed such that the first 2^+ states in ^{106}Cd and ^{106}Pd be reproduced, while for the particle-particle interactions they were fixed to unity.

3.1 Two-neutrino modes

We calculated first the $M_{\mathrm{GT}}^{2\nu}$ (eq. (2.12)) as a function of g_{pp} using the two s.p. bases. We remind that the expression (2.12) is derived for the $\beta^+\beta^+$ mode in the approximation that the four emitting leptons share equally their kinetic energy. For the other two decay modes (*i.e.* β^+/EC and EC/EC) we have to adapt this approximation, since in

Table 1. The matrix elements and the half-lives for the 2νEC/EC, $2\nu\beta^+$/EC and $2\nu\beta^+\beta^+$ decay modes. The NME are calculated with $g_{\mathrm{pp}} = 0.9$ and the half-lives with the phase space factors taken from table 2. The two sets of numbers of refs. [8] and [30] represent calculations performed with two different s.p. bases, as is explained in those references. (l) and (s) denote the calculations performed with the larger and smaller basis, respectively.

	Exp	QRPA [28]	QRPA [30]	RQRPA [8]	SQRPA
$M^{2\nu}_{\mathrm{GT}}$	< 8.7	0.27	0.84	0.55	0.61 (l)
			0.78	0.56	0.57 (s)
$T^{2\nu}_{1/2}$(EC/EC)	> 2.6 (17) [35]	8.7 (20)	9.0 (19)	2.1 (20)	2.6(20) (l)
			1.0 (20)	2.0 (20)	1.96 (20) (s)
$T^{2\nu}_{1/2}(\beta^+/\mathrm{EC})$	> 6.6 (18) [30]	4.1 (21)	7.1 (20)	1.7 (20)	1.36 (21) (l)
	> 4.1 (20) [31]		8.2 (20)	1.6 (21)	1.56 (21) (s)
$T^{2\nu}_{1/2}(\beta^+\beta^+)$	> 9.2 (17) [29]	4.6 (26)	2.84 (25)	6.62 (25)	5.38 (25) (l)
			3.3 (25)	6.39 (25)	6.16 (25) (s)

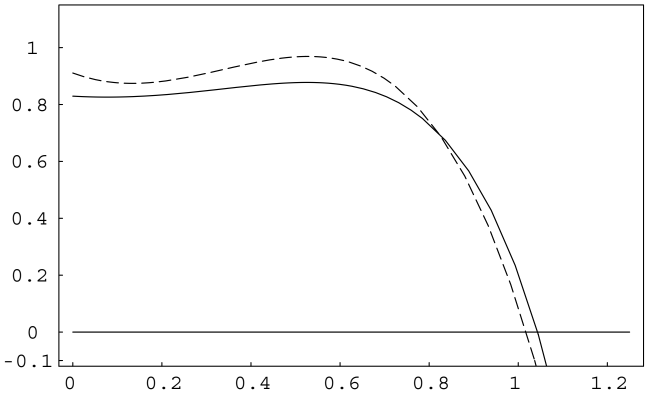

Fig. 1. The $M^{2\nu}$ matrix elements *versus* g_{pp} calculated with SQRPA methods for the β^+/EC decay of ^{106}Cd. The dashed line and the solid line correspond to the calculations performed with the large (l) and small (s) s.p. basis, respectively.

Table 2. The integrated phase space factors F^a for the 2ν and 0ν decay modes used in the calculation of the coefficients C_{xy}. They are taken from ref. [33]. The kind of the decay mode is indicated and the numbers in parenthesis represent powers of ten. The value of the phase space factor for the 2νEC/EC mode is 1.573(−20).

F^a (y^{-1})	$\beta^+\beta^+$	β^+/EC
F_1	2.589 (−18)	3.717 (−17)
F_2	2.825 (−19)	6.407 (−16)
F_3	2.827 (−19)	2.408 (−16)
F_4	3.961 (−19)	2.572 (−18)
F_5	−1.316 (−16)	1.251 (−15)
F_6	7.971 (−16)	7.908 (−15)
F_7	−4.110 (−14)	3.402 (−13)
F_8	3.364 (−15)	2.690 (−14)
F_9	1.256 (−13)	1.075 (−12)
$F^{2\nu}$	4.991 (−24)	1.97 (−21)

these cases we have only three and two free leptons, respectively, in the final states. Thus, the denominator in (2.12) for these decay modes was chosen as follows: for the mixed mode, β^+/EC, where we have only three free leptons in the final states, we replaced the energy of a free electron by the energy of a bound electron and then we assumed that the available energy is shared equally between the three free leptons. For the EC/EC mode we replaced the energies of both free electrons by their bound energies in atomic shells and we assumed that the two neutrinos share equally the available energy. The denominator will be smaller and thus the values of the NME are enhanced for these decay modes. However, the large differences in their half-lives are coming mainly from the very different phase space factors and not from these differences in the NME.

Since the behaviour of the NME is similar for the three processes investigated, we show in fig. 1 only the NME for the β^+/EC process as function of g_{pp}. One can see that the results display a strong dependence on this parameter: after an almost constant value up to $g_{\mathrm{pp}} = 0.75$, they drop steeply and vanish around $g_{\mathrm{pp}} = 1.05$. By contrast, the differences between the calculations performed with the two s.p. bases are not large and the behaviour of the two curves is similar, but the NME obtained with the large basis are more stable. Generally, the results resemble to the corresponding ones obtained with pnQRPA and RQRPA in refs. [20,28]. However, the more instable behaviour of the NME found in these references even at smaller values of g_{pp} is not so pronounced in our calculations. The reason for such a strong dependence on g_{pp} might be the dominant contribution of the first 1^+ state to the NME found in our calculations. Indeed, we compared the calculation of the NME by i) including the contribution of only the first 1^+ state in ^{106}Ag (which is its g.s.) and ii) including the contribution from the whole set of intermediate 1^+ states and we found a rather small difference between the values, at the same $g_{\mathrm{pp}} = 0.9$ (i): 0.83, ii): 0.61). The strong dependence on the model parameter and the non-existence, up to now, of definite experimental half-lives for these decays make the prediction of the NME very difficult. Thus, for the calculation of the half-lives g_{pp} must be fixed from other experimental quantities, for instance from single β decays connected to the involved nuclei, as was proposed already in [3]. We used the value $g_{\mathrm{pp}} = 0.9$

Table 3. The nine NME which enter in the $T^{0\nu}_{1/2}$ half-lives. The values of the first two rows represent the calculations of this work performed with the smaller (s) and larger (l) basis, while those of the third row are taken from ref. [28].

	M_{GT}	M_F	$M_{\mathrm{GT}\omega}$	$M_{F\omega}$	$M_{\mathrm{GT}q}$	M_{Fq}	M_T	M_P	M_R
(s)	5.99	−2.18	5.65	−2.01	4.26	−1.88	−0.67	2.46	7.45
(l)	5.73	−2.12	5.21	−1.94	4.15	−1.27	−0.62	2.34	7.06
[28]	3.34	−1.22	3.14	−1.09	2.35	−1.05	−0.38	1.43	4.10

Table 4. The coefficients C_{xy} calculated with the NME from table 2 and the integrated space phase factors from table 1. The values in the first two rows of each line represent calculations of this work performed with (s) and (l) basis, respectively, while those in the third row represent calculations of the authors of ref. [28].

	C_{mm}	$C_{m\eta}$	$C_{m\lambda}$	$C_{\eta\eta}$	$C_{\lambda\lambda}$	$C_{\eta\lambda}$
$\beta^+\beta^+$	(s) 1.7 (−16)	−5.1 (−14)	−8.5 (−18)	7.2 (−12)	1.2 (−17)	−5.2 (−18)
	(l) 1.6 (−16)	−4.7 (−14)	−2.3 (−18)	6.9 (−12)	9.3 (−18)	−6.2 (−18)
	5.4 (−17)	−1.6 (−14)	−2.6 (−18)	2.4 (−12)	4.4 (−18)	−1.6 (−18)
β^+/EC	(s) 2.5 (−15)	5.1 (−13)	1.5 (−14)	6.6 (−11)	3.5 (−14)	−2.1 (−14)
	(l) 2.3 (−15)	4.6 (−13)	1.3 (−14)	5.9 (−11)	2.9 (−14)	−1.8 (−14)
	7.7 (−16)	1.0 (−13)	4.5 (−15)	2.0 (−11)	1.1 (−14)	−6.6 (−15)

which fits best the experimental $\log ft$ corresponding to the (β^+, EC) and β^- decays of ^{106}Ag to the ^{106}Pd and ^{106}Cd, respectively (g.s.-to-g.s. transitions).

Then, we calculated the half-lives for the three decay modes with emission of neutrinos. For this we used the phase space factors from ref. [33] calculated with relativistic electron wave functions.

In table 1 we give our predicted half-lives for the decay modes indicated together with the NME. For comparison, we also give the results of other works. One can see that, for the β^+/EC decay mode, the predicted half-lives are not far from the present experimental limits, which is encouraging for the continuation of the measurements to detect such a decay.

Finally, we checked the Ikeda sum rule within SQRPA and we found it is conserved with good accuracy. The deviations from the exact fulfillment, both for the initial and final nuclei, were within 2-3 percent. Here, we would like to mention that our higher-order corrections to the pnQRPA introduced by boson expansion of A and B operators affect both the wave functions and density-type operators. As was described in our previous papers (for instance, [7] and [14]), the introduction of such corrections does not affect much the structure of the beta transition strengths (only the β^+ transitions are a bit redistributed as compared with their pnQRPA image). So, while the ISR is fulfilled with good accuracy, also the Fermi beta-type transitions remain mostly concentrate in the IAS.

3.2 Neutrinoless modes

We performed calculations of the NME for the $0\nu\beta^+\beta^+$ and $0\nu\beta^+$/EC decay modes using the two s.p. bases.

The results are summarized in tables 2-4. In table 2 we give the phase factors used for computing the C^{α}_{xy}-coefficients entering the formulae for the neutrinoless $\beta\beta$ decay half-lives. They were taken from ref. [33]. In table 3 we give the NME relevant for the neutrinoless decay modes both for the mass mechanism and the right-handed currents and for both the s.p. bases. For comparison, we also give in the third row the values of the NME obtained by the authors of ref. [28]. One can see that our values are systematically larger by factors of 1.5-2 than those from ref. [28]. One also observes that the NME obtained with the two s.p. bases are close to each other, so the enlargement of the bases seems to have small influence on their values.

In table 4 we give the values of the C_{xy}-coefficients which enter into the neutrinoless decay modes half-lives. The differences in the NME reflected in the values of these coefficients are about a factor of 3-4 compared to the values obtained in [28]. The enhancement of the coefficients $C_{\lambda\lambda}$, $C_{m\lambda}$, $C_{\eta\lambda}$ for the β^+/EC decay mode is significant as compared to the corresponding coefficients for the $\beta^+\beta^+$ decay mode (3-4 orders of magnitude), as was already discussed in [28]. This difference is coming mainly from the enhancement of the F_2 and F_3 phase space factors in the case of the β^+/EC decay mode, which are dominant in the calculation of the above-mentioned coefficients.

In order to see the relative importance of the mass and RH-current mechanisms, one can calculate the magnitude of the different terms appearing in the expression of the neutrinoless half-lives (2.14). If we assume that the $0\nu\beta^+\beta^+$ decay occurs only by the mass mechanism and use the value of the neutrino mass parameter reported recently in ref. [36] (*i.e.* $\langle m_\nu\rangle \approx 0.39\,\mathrm{eV}$), one gets $T^{0\nu}_{1/2}(\beta^+\beta^+) \sim 10^{28}$ y and $T^{0\nu}_{1/2}(\beta^+/\mathrm{EC}) \sim 7\times10^{26}$ y. The same order of magnitude is obtained with the following RH parameters: i) for the $\beta^+\beta^+$ mode: $\langle\lambda\rangle \sim 3\times10^{-6}$, $\langle\eta\rangle \sim 3.8\times10^{-9}$ and ii) for the β+/EC mode: $\langle\lambda\rangle \sim 2.1 \sim 10^{-7}$, $\langle\eta\rangle \sim 4.8\times10^{-9}$. So, from $0\nu\beta^+$/EC experiments one can extract more stringent limits for the λ-parameter than from the $0\nu\beta^+\beta^+$ ones, if there exists experimental evidence for the neutrinoless $\beta\beta$ decay. This is why, an experimental analysis of this decay mode can be of interest for obtaining additional information about the dominant mechanism governing the $\beta\beta$ decay (see also [28]).

4 Conclusions

We calculated the nuclear matrix elements and half-lives for the neutrino-emitting modes $\beta^+\beta^+$, β^+/EC and EC/EC, and for neutrinoless $\beta^+\beta^+$ and β^+/EC decay modes with the SQRPA method and using two s.p. bases in the case of ^{106}Cd. Although this method has extensively been used in $\beta^-\beta^-$ decay calculations [23–25], it is now for the first time here used for $\beta^+\beta^+$ decay calculations.

For the neutrino-emitting decay modes we found a strong dependence of the NME on $g_{\rm pp}$, while the results depend weakly on the size of the s.p. basis used. One of the reasons for such a strong dependence on $g_{\rm pp}$ might be the dominant contribution of the 1^+ state to the NME, which was observed in our calculations. Fixing $g_{\rm pp}$ such that the corresponding experimental $\log ft$ for the single β decay is reproduced best, and using the space phase factors from ref. [33], we got half-lives for the $\beta^+\beta^+$ and β^+/EC decay of the order of 10^{21} y, which are not far from the present experimental limits. This may be encouraging for new planned experiments. Also, checking the ISM we found deviations from the exact fulfillment, for both the initial and final nuclei, which are within a few percent.

Further, we performed calculations of the NME for the $0\nu\beta^+\beta^+$ and $0\nu\beta^+$/EC decay modes using two s.p. bases for both the mass and the RH-current mechanisms. The values we obtained are slightly larger, by factors of 1.5-2, than those from [28] and they do not depend much on the s.p. basis used. Then, we computed the C_{xy}-coefficients entering the half-life formulae and confirmed the enhancement of $C_{\lambda\lambda}$, $C_{m\lambda}$, $C_{\eta\lambda}$ in the case of the β^+/EC decay mode as compared with the corresponding coefficients for the $\beta^+\beta^+$ decay mode by 3-4 orders of magnitude [28]. Using the recently reported value for the neutrino mass parameter (*i.e.* $\langle m_\nu\rangle \approx 0.39$ eV [36]), we got $T^{0\nu}_{1/2}(\beta^+\beta^+) \sim 10^{28}$ y and $T^{0\nu}_{1/2}(\beta^+/\mathrm{EC}) \sim 7\times10^{26}$ y. With these values we extracted the RH parameters from our calculations and found, in the case of the $0\nu\beta$/EC decay, a value for the λ-parameter which is about one order of magnitude smaller than the one extracted from $0\nu\beta^+\beta^+$. This difference comes from the enhancement of the $C_{m,\eta,\lambda\lambda}$-coefficients and demonstrates the larger sensitivity of the neutrinoless mixed mode experiments on the RH-current contributions to the $\beta^+\beta^+$ decay.

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Appendix A.

For the overlap factors which appear in the expression of the NME we used the following expressions:

$$\langle 1_{k_1}|1_{k'_1}\rangle = \sum_{j_pj_n}\Big(X^1_{k_1}(j_pj_n)\bar{X}^1_{k'_1}(j_pj_n) - Y^1_{k_1}(j_pj_n)\bar{Y}^1_{k'_1}(j_pj_n)\Big)\,, \tag{A.1}$$

$$\langle(1_{k_1}2_{k_2})1|(1_{k'_1}2_{k'_2})1\rangle = \sum_{j_pj_n}\Big(X^1_{k_1}(j_pj_n)\bar{X}^1_{k'_1}(j_pj_n) - Y^1_{k_1}(j_pj_n)\bar{Y}^1_{k'_1}(j_pj_n)\Big) \times\sum_{j_\tau j'_\tau}\Big(X^2_{k_2}(j_\tau j'_\tau)\bar{X}^2_{k'_2}(j_\tau j'_\tau) - Y^2_{k_2}(j_\tau j'_\tau)\bar{Y}^2_{k'_2}(j_\tau j'_\tau)\Big)\,, \tag{A.2}$$

where $\tau = p$ or n.

The nine NME which appear in (2.15) are:

$$M_\alpha = \sum\langle 0^+_f||\tau^+_1\tau^+_2 O^\alpha_{12}||0^+_i\rangle\,, \tag{A.3}$$

where the two-body transition operators O^α_{12} have the following definitions:

$$O^{\rm GT}_{12} = \sigma_1\sigma_2 H_m(r)\,, \qquad O^{\rm GT\omega}_{12} = \sigma_1\sigma_2 H_\omega(r)\,,$$
$$O^{\rm GTq}_{12} = \sigma_1\sigma_2\frac{r}{R}H_q(r)\,, \qquad O^F_{12} = H_m(r)\left(\frac{g_V}{g_A}\right)^2,$$
$$O^{F\omega}_{12} = H_\omega(r)\left(\frac{g_V}{g_A}\right)^2, \qquad O^{Fq}_{12} = \frac{r}{R}H_q(r)\left(\frac{g_V}{g_A}\right)^2,$$
$$O^T_{12} = \left[(\sigma_1\hat{\mathbf{r}})(\sigma_2\hat{\mathbf{r}}) - \frac{1}{3}\sigma_1\sigma_2\right]\frac{r}{R}H_q(r)\,,$$
$$O^P_{12} = i(\sigma_1-\sigma_2)\left(\hat{\mathbf{r}}\times\frac{\mathbf{r}_+}{R}\right)H_q(r)\left(\frac{g_V}{g_A}\right),$$
$$O^R_{12} = \sigma_1\sigma_2 H_q(r)\frac{\mu_\beta}{3}\left(\frac{g_V}{g_A}\right), \tag{A.4}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$, $\mathbf{r}_+ = (\mathbf{r}_1+\mathbf{r}_2)/2$ and R is the nuclear radius which is introduced to make the matrix elements dimensionless. The neutrino potentials defined by integrals over the neutrino momentum are also dimensionless and have the following expressions [10]:

$$H_m(r) = \frac{2R}{\pi}\frac{1}{r}\int_0^\infty \frac{q\sin(qr)}{\omega(\omega+\bar{E})}\mathrm{d}q\,,$$
$$H_\omega(r) = \frac{2R}{\pi}\frac{1}{r}\int_0^\infty \frac{q\sin(qr)}{\omega(\omega+\bar{E})^2}\mathrm{d}q\,,$$
$$H_q(r) = -R\frac{\mathrm{d}}{\mathrm{d}r}H_m(r)\,,$$
$$H_R(r) = -\frac{R}{M}\frac{\mathrm{d}^2}{\mathrm{d}\mathbf{r}^2}H_m(r)\,, \tag{A.5}$$

were $\omega = \sqrt{q^2+m^2_\nu}$ and $\bar{E}$ is the average energy of intermediate nuclear states. The two-body short-range correlations between nucleons are taken into account by multiplying the two-particle wave function by the correlation function:

$$1 - f(r) = 1 - e^{-ar^2}(1-br^2) \tag{A.6}$$

with $a = 1.1\,\mathrm{fm}^{-2}$ and $b = 0.68\,\mathrm{fm}^{-2}$. The finite nucleon size effects are introduced by the monopole form factors in momentum space [37]:

$$g_{V/A} \to g_{V/A}\left(\frac{\Lambda^2}{\Lambda^2+q^2}\right)^2 \qquad \text{(A.7)}$$

with $\Lambda = 850\,\mathrm{MeV}$.

The NME for the neutrinoless $\beta\beta$ decay can be expressed as a sum of products of one-body transition densities and transition matrix elements for two-particle states:

$$M_a = \sum_{p_1,p_2,n_1,n_2,J',J,\pi} Z(p_1p_2,n_1n_2;J'J^\pi) \times \langle p_1p_2;J'|\tau_1^+\tau_2^+O_{12}^a|n_1n_2;J'\rangle\,, \qquad \text{(A.8)}$$

where in our case

$$Z(p_1p_2,n_1n_2;J',1^+) =$$

$$3(-)^{p_2+n_1+J'+J}(2J'+1)W(p_2p_1n_2n_1;J'1)$$

$$\times\langle 0_f^+||\widetilde{[c_{p_2}^\dagger\tilde{c}_{n_2}]_1}||1_k^+\rangle\langle 1_k^+|1_{k'}^+\rangle\langle 1_{k'}^+||[c_{p_1}^\dagger\tilde{c}_{n_1}]_1||0_i^+\rangle. \qquad \text{(A.9)}$$

The one-body densities of the single β-type operators are calculated within our model space (2.1) and their expressions are similar to the (2.10), (2.11) ones.

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